For $2 < n+1 < m$, $Ur\mathfrak{Nr}_n\mathbf{CA}_m$ is not finitely axiomatizable

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Abstract . We show that for $2 < n+1 < m$, the class $\mathfrak{Nr}_n\mathbf{CA}_m$ is psuedo elementary, whose elementary that is not finitely axiomatizable.

The class of neat reducts has been extensively studied by the author, Andréka, Németi, Hirsch, Hodkinson, Ferenzci and others. In this note we show that for $1 < n < m$, the class $\mathfrak{Nr}_n\mathbf{CA}_m$ is psuedo-elementary (it is known that it is not closed under ultraroots Ur [3]), and that for $2 < n+1 < m < \omega$, the class $Ur\mathfrak{Nr}_nCA_m$ is not finitely axiomatizabe. For our first result we use a defining theory in two sorts when both n and m are finite, three sorts when n is finite and m is infinite, and four sorts when both $m$and n are infinite. For our second result we use Monk-like algebras constructed by Robin Hirsch.

Theorem 0.1. *Let $1 < n < m$, then the class $\mathfrak{Nr}_n\mathbf{CA}_m$ is pseudo-elementary, but is not elementary. Furthermore, $EL\mathfrak{Nr}_n\mathbf{CA}_m$is recursively enumerable, and for $n > 2$, and $m \geq 2$, $EL\mathfrak{Nr}_n\mathbf{CA}_m$ is not finitely axiomatizable.*

Proof.

(1) For $n < m < \omega$, the charactersation is easy. One defines the class $\mathfrak{Nr}_n\mathbf{CA}_m$ in a two sorted language. The first sort for the n dimensional cylindric algebra the second for the m dimensinal cylindric algebra. The signature of the defining theory includes an injective function I from sort one to sort two and includes a sentence requiring that I respects the operations and a sentence saying that an element of the second sort say y satisfies $\bigvee_{n\leq i<m} c_i y = y$, iff there exists x of sort one such that $y = I(x)$ so that I is a bijection.

Assume that n is still finite, we first show that for any infinite α, $\mathfrak{Nr}_n\mathbf{CA}_\omega = \mathfrak{Nr}_n\mathbf{CA}_\alpha$. Let $\mathfrak{A} \in \mathfrak{Nr}_n\mathbf{CA}_\omega$, so that $\mathfrak{A} = \mathfrak{Nr}_n\mathfrak{B}'$, $\mathfrak{B}' \in \mathbf{CA}_\omega$. Let $\mathfrak{B} =$

$\mathfrak{Sg}^{\mathfrak{B}'}A$. Then $\mathfrak{B}\in \mathbf{Lf}_\omega$, and $\mathfrak{A}=\mathfrak{Nr}_n\mathfrak{B}$. But $\mathbf{Lf}_\omega=\mathfrak{Nr}_\omega\mathbf{Lf}_\alpha$ and we are done. To show that $\mathfrak{Nr}_n\mathbf{CA}_\omega\subseteq \mathfrak{Nr}_n\mathbf{RCA}_\omega$, let $\mathfrak{A}\in \mathfrak{Nr}_n\mathbf{CA}_\omega$, then by the above argument there exists then $\mathfrak{B}\in \mathbf{Lf}_\omega$ such that $\mathfrak{A}=\mathfrak{Nr}_n\mathfrak{B}$. by $\mathbf{Lf}_\omega\subseteq \mathbf{RCA}_\omega$, we are done.

It is known that class $\mathfrak{Nr}_n\mathbf{CA}_\omega$ is not elementary. In fact, there is an algebra $\mathfrak{A}\in \mathfrak{Nr}_n\mathbf{CA}_\omega$ having a complete subalgebra $\mathfrak{B}$, and $\mathfrak{B}\notin \mathfrak{Nr}_n\mathbf{CA}_{n+1}$, this will be proved below.

Now assume that m is infinite. Here if y is in the n dimensional cylindric algebra then we cannot express $c_i=y$ for all $i\in \omega\sim n$, like we did when m is finite, so we have to think differently.

To show that it is pseudo-elementary, we use a three sorted defining theory, with one sort for a cylindric algebra of dimension n (c), the second sort for the Boolean reduct of a cylindric algebra (b) and the third sort for a set of dimensions (δ). We use superscripts n, b, δ for variables and functions to indicate that the variable, or the returned value of the function, is of the sort of the cylindric algebra of dimension n, the Boolean part of the cylindric algebra or the dimension set, respectively. The signature includes dimension sort constants i^δ for each $i<\omega$ to represent the dimensions. The defining theory for $\mathfrak{Nr}_n\mathbf{CA}_\omega$ incudes sentences demanding that the consatnts i^δ for $i<\omega$ are distinct and that the last two sorts define a cylindric algenra of dimension ω. For example the sentence

$$\forall x^\delta, y^\delta, z^\delta(d^b(x^\delta,y^\delta)=c^b(z^\delta, d^b(x^\delta,z^\delta).d^b(z^\delta,y^\delta)))$$

represents the cylindric algebra axiom $\mathsf{d}_{ij}=\mathsf{c}_k(\mathsf{d}_{ik}.\mathsf{d}_{kj})$ for all $i,j,k<\omega$. We have have a function I^b from sort c to sort b and sentences requiring that I^b be injective and to respect the n dimensional cylindric operations as follows: for all x^r

$$I^b(\mathsf{d}_{ij})=d^b(i^\delta,j^\delta)$$
$$I^b(\mathsf{c}_ix^r)=\mathsf{c}_i^b(I^b(x)).$$

Finally we require that I^b maps onto the set of n dimensional elements

$$\forall y^b((\forall z^\delta(z^\delta\neq 0^\delta,\dots(n-1)^\delta\to c^b(z^\delta,y^b)=y^b))\leftrightarrow \exists x^r(y^b=I^b(x^r))).$$

In this case we need a fourth sort. We leave the details to the reader.

In all cases, it is clear that any algebra of the right type is the first sort of a model of this theory. Conversely, a model for this theory will consist of an n dimensional cylindric algebra type (sort c), and a cylindric algebra whose dimension is the cardinality of the δ-sorted elements, which is at least $|m|$. Thus the first sort of this model must be a neat reduct.

(2) For $\mathfrak{A} \in \mathbf{CA}_n$, $\mathfrak{Rd}_3\mathfrak{A}$ denotes the $\mathbf{CA}_3$ obtained from $\mathfrak{A}$ by discarding all operations indexed by indices in $n \sim 3$. $\mathbf{Df}_n$ denotes the class of diagonal free cylindric algebras. $\mathfrak{Rd}_{df}\mathfrak{A}$ denotes the $\mathbf{Df}_n$ obtained from $\mathfrak{A}$ by deleting all diagonal elements. To prove the non-finite axiomatizability result we use Monk's algebras given above.

For $3 \leq n, i < \omega$, with $n - 1 \leq i$, $\mathfrak{C}_{n,i}$ denotes the $\mathbf{CA}_n$ associated with the cylindric atom structure as defined on p. 95 of [1]. Then by [1, 3.2.79] for $3 \leq n$, and $j < \omega$, $\mathfrak{Rd}_3\mathfrak{C}_{n,n+j}$ can be neatly embedded in a $\mathbf{CA}_{3+j+1}$. (1) By [1, 3.2.84]) we have for every $j \in \omega$, there is an $3 \leq n$ such that $\mathfrak{Rd}_{df}\mathfrak{Rd}_3\mathcal{C}_{n,n+j}$ is a non-representable $\mathbf{Df}_3$. (2) Now suppose $m \in \omega$. By (2), choose $j \in \omega \sim 3$ so that $\mathfrak{Rd}_{df}\mathfrak{Rd}_3\mathfrak{C}_{j,j+m+n-4}$ is a non-representable $\mathbf{Df}_3$. By (1) we have $\mathfrak{Rd}_{df}\mathfrak{Rd}_3\mathfrak{C}_{j,j+m+n-4} \subseteq \mathfrak{Nr}_3\mathfrak{B}_m$, for some $\mathfrak{B} \in \mathbf{CA}_{n+m}$. Put $\mathfrak{A}_m = \mathfrak{Nr}_n\mathfrak{B}_m$. $\mathfrak{Rd}_{df}\mathfrak{A}_m$ is not representable, a friotri, $\mathfrak{A}_m \notin \mathbf{RCA}_n$, for else its $\mathbf{Df}$ reduct would be representable. Therefore $\mathfrak{A}_m \notin EL\mathfrak{Nr}_n\mathbf{CA}_\omega$. Now let $\mathfrak{C}_m$ be an algebra similar to $\mathbf{CA}_\omega$'s such that $\mathfrak{B}_m = \mathfrak{Rd}_{n+m}\mathfrak{C}_m$. Then $\mathfrak{A}_m = \mathfrak{Nr}_n\mathfrak{C}_m$. Let F be a non-principal ultrafilter on ω. Then

$$\prod_{m\in\omega}\mathfrak{A}_m/F = \prod_{m\in\omega}(\mathfrak{Nr}_n\mathfrak{C}_m)/F = \mathfrak{Nr}_n(\prod_{m\in\omega}\mathfrak{C}_m/F)$$

But $\prod_{m\in\omega}\mathfrak{C}_m/F \in \mathbf{CA}_\omega$. Hence $\mathbf{CA}_n \sim El\mathfrak{Nr}_n\mathbf{CA}_\omega$ is not closed under ultraproducts. It follows that the latter class is not finitely axiomatizable. In [?] it is proved that for $1 < \alpha < \beta$, $El\mathfrak{Nr}_\alpha\mathbf{CA}_\beta \subset S\mathfrak{Nr}_\alpha\mathbf{CA}_\beta$.

(3) This follows from the folowing known fact (a result of Hirsch and Sayed ahmed, submitted for publication) For $3 \leq m < n < \omega$, there is m dimensional algebra $\mathfrak{C}(m, n, r)$ such that

(1) $\mathfrak{C}(m, n, r) \in \mathfrak{Nr}_m\mathbf{CA}_n$

(2) $\mathfrak{C}(m, n, r) \notin S\mathfrak{Nr}_m\mathbf{CA}_{n+1}$

(3) $\prod_{r\in\omega}\mathfrak{C}(m, n, r) \in El\mathfrak{Nr}_m\mathbf{CA}_n$

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From the above proof it follows that

Corollary 0.2. *Let K be any class such that $\mathfrak{Nr}_n\mathbf{CA}_\omega \subseteq K \subseteq \mathbf{RCA}_n$. Then ELK is not finitely axiomatizable*

Theorem 0.3. *For α infinite, and $k \in \omega$ there is $\mathfrak{A} \in \mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+k} \sim S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+k+1}$*

Proof. Let $\mathfrak{C}(m, n, r)$ be as consrtucted above, then we also have: For $m < n$ and $k \geq 1$, there exists $x_n \in \mathfrak{C}(n, n+k, r)$ such that $\mathfrak{C}(m, m+k, r) \cong \mathfrak{Rl}_x\mathfrak{C}(n, n+k, r)$. Let α be an infinite ordinal, let X be any finite subset of α,

let $I = \{\Gamma : X \subseteq \Gamma \subseteq \alpha,\ |\Gamma| < \omega\}$. For each $\Gamma \in I$ let $M_\Gamma = \{\Delta \in I : \Delta \supseteq \Gamma\}$ and let F be any ultrafilter over I such that for all $\Gamma \in I$ we have $M_\Gamma \in F$ (such an ultrafilter exists because $M_{\Gamma_1} \cap M_{\Gamma_2} = M_{\Gamma_1 \cup \Gamma_2}$). For each $\Gamma \in I$ let ρ_Γ be a bijection from $|\Gamma|$ onto Γ. For each $\Gamma \in I$ let $\mathcal{A}_\Gamma, \mathcal{B}_\Gamma$ be $\mathbf{CA}_\alpha$-type algebras. For each $\Gamma \in I$ we have $\mathfrak{Rd}^{\rho_\Gamma}\mathcal{A}_\Gamma = \mathfrak{Rd}^{\rho_\Gamma}\mathcal{B}_\Gamma$ then $\Pi_{\Gamma/F}\mathcal{A}_\Gamma = \Pi_{\Gamma/F}\mathcal{B}_\Gamma$.

Furthermore, if $\mathfrak{Rd}^{\rho_\Gamma}\mathcal{A}_\Gamma \in \mathbf{CA}_{|\Gamma|}$, for each $\Gamma \in I$ then $\Pi_{\Gamma/F}\mathcal{A}_\Gamma \in \mathbf{CA}_\alpha$.

Let $k \in \omega$. Let α be an infinite ordinal. Then $S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+k+1} \subset S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+k}$. Let $r \in \omega$. Let $I = \{\Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega\}$. For each $\Gamma \in I$, let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$, and let F be an ultrafilter on I such that $\forall \Gamma \in I,\ M_\Gamma \in F$. For each $\Gamma \in I$, let ρ_Γ be a one to one function from $|\Gamma|$ onto Γ. Let $\mathcal{C}^r_\Gamma$ be an algebra similar to $\mathbf{CA}_\alpha$ such that

$$\mathfrak{Rd}^{\rho_\Gamma}\mathcal{C}^r_\Gamma = \mathcal{C}(|\Gamma|, |\Gamma| + k, r).$$

Let

$$\mathfrak{B}^r = \prod_{\Gamma/F \in I} \mathcal{C}^r_\Gamma.$$

We will prove that

1. $\mathfrak{B}^r \in \mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+k}$ and
2. $\mathfrak{B}^r \notin S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+k+1}$.

The theorem will follow, since $\mathfrak{Rd}_{\mathbf{CA}}\mathfrak{B}^r \in S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+k} \setminus S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+k+1}$.

For the first part, for each $\Gamma \in I$ we know that $\mathcal{C}(|\Gamma|+k, |\Gamma|+k, r) \in \mathbf{CA}_{|\Gamma|+k}$ and $\mathfrak{Nr}_{|\Gamma|}\mathcal{C}(|\Gamma|+k, |\Gamma|+k, r) \cong \mathcal{C}(|\Gamma|, |\Gamma|+k, r)$. Let σ_Γ be a one to one function $(|\Gamma|+k) \to (\alpha+k)$ such that $\rho_\Gamma \subseteq \sigma_\Gamma$ and $\sigma_\Gamma(|\Gamma|+i) = \alpha+i$ for every $i < k$. Let $\mathcal{A}_\Gamma$ be an algebra similar to a $\mathbf{CA}_{\alpha+k}$ such that $\mathfrak{Rd}^{\sigma_\Gamma}\mathcal{A}_\Gamma = \mathcal{C}(|\Gamma|+k, |\Gamma|+k, r)$. By the second part with $\alpha + k$ in place of α, $m \cup \{\alpha + i : i < k\}$ in place of X, $\{\Gamma \subseteq \alpha + k : |\Gamma| < \omega,\ X \subseteq \Gamma\}$ in place of I, and with σ_Γ in place of ρ_Γ, we know that $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \in \mathbf{CA}_{\alpha+k}$.

We prove that $\mathfrak{B}^r \subseteq \mathfrak{Nr}_\alpha\Pi_{\Gamma/F}\mathcal{A}_\Gamma$. Recall that $\mathfrak{B}^r = \Pi_{\Gamma/F}\mathcal{C}^r_\Gamma$ and note that $C^r_\Gamma \subseteq A_\Gamma$ (the base of C^r_Γ is $\mathcal{C}(|\Gamma|, |\Gamma| + k, r)$, the base of A_Γ is $\mathcal{C}(|\Gamma| + k, |\Gamma| + k, r)$). So, for each $\Gamma \in I$,

$$\begin{aligned}
\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}^r_\Gamma &= \mathcal{C}((|\Gamma|, |\Gamma| + k, r) \\
&\cong \mathfrak{Nr}_{|\Gamma|}\mathcal{C}(|\Gamma| + k, |\Gamma| + k, r) \\
&= \mathfrak{Nr}_{|\Gamma|}\mathfrak{Rd}^{\sigma_\Gamma}\mathfrak{A}_\Gamma \\
&= \mathfrak{Rd}^{\sigma_\Gamma}\mathfrak{Nr}_\Gamma\mathfrak{A}_\Gamma \\
&= \mathfrak{Rd}^{\rho_\Gamma}\mathfrak{Nr}_\Gamma\mathfrak{A}_\Gamma
\end{aligned}$$

By the first part of the first part we deduce that $\Pi_{\Gamma/F}\mathfrak{C}^r_\Gamma \cong \Pi_{\Gamma/F}\mathfrak{Nr}_\Gamma\mathfrak{A}_\Gamma = \mathfrak{Nr}_\alpha\Pi_{\Gamma/F}\mathfrak{A}_\Gamma$, proving (1).

Now we prove (2). For this assume, seeking a contradiction, that $\mathfrak{B}^r \in S\mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+k+1}$, $\mathfrak{B}^r \subseteq \mathfrak{Nr}_\alpha \mathcal{C}$, where $\mathcal{C} \in \mathbf{CA}_{\alpha+k+1}$. Let $3 \leq m < \omega$ and $\lambda : m+k+1 \to \alpha+k+1$ be the function defined by $\lambda(i) = i$ for $i < m$ and $\lambda(m+i) = \alpha + i$ for $i < k+1$. Then $\mathfrak{Rd}^\lambda(\mathcal{C}) \in \mathbf{CA}_{m+k+1}$ and $\mathfrak{Rd}_m \mathfrak{B}^r \subseteq \mathfrak{Nr}_m \mathfrak{Rd}^\lambda(\mathcal{C})$. For each $\Gamma \in I$, let $I_{|\Gamma|}$ be an isomorphism

$$\mathcal{C}(m, m+k, r) \cong \mathfrak{Rl}_{x_{|\Gamma|}} \mathfrak{Rd}_m \mathcal{C}(|\Gamma|, |\Gamma+k|, r).$$

Let $x = (x_{|\Gamma|} : \Gamma)/F$ and let $\iota(b) = (I_{|\Gamma|} b : \Gamma)/F$ for $b \in \mathcal{C}(m, m+k, r)$. Then ι is an isomorphism from $\mathcal{C}(m, m+k, r)$ into $\mathfrak{Rl}_x \mathfrak{Rd}_m \mathfrak{B}^r$. Then $\mathfrak{Rl}_x \mathfrak{Rd}_m \mathfrak{B}^r \in S\mathfrak{Nr}_m \mathbf{CA}_{m+k+1}$. It follows that $\mathcal{C}(m, m+k, r) \in S\mathfrak{Nr}_m \mathbf{CA}_{m+k+1}$ which is a contradiction and we are done. $\square$

References

[1] L. Henkin, J.D. Monk and A.Tarski, *Cylindric Algebras Part I.* North Holland, 1971.

[2] L. Henkin, J.D. Monk and A.Tarski, *Cylindric Algebras Part II.* North Holland, 1985.

[3] T. Sayed Ahmed *A model-theoretic solution to a problem of Tarski* Math Logic Quarterly (48)(2002) 343-355